CentraleSupélec
Accelerated Engineering Degree Programme
Master's level

September 2018

Written test - Mathematics

2 hours

A standard -non scientific- language dictionary is authorized. Please make sure to have it checked by the staff. Documents, electronic devices and calculators are not allowed.

Exercise 1 Let E be a normed vector space. Let A be a convex subset of E.

Q. 1.1 *Show that the interior of* A *is also a convex subset of* E.

Q. 1.2 *Show that the closure of* A *is also a convex subset of* E.

Q. 1.3 *Let* E *be the space of continuous functions on* $[0,1]$. *Does the sequence* $f_n : x \mapsto x^n$ *converge pointwise ? Uniformly ? Give the limit if it exists. Prove your claims.*

Exercise 2 Let P be a polynomial with real coefficients, such that, for all $x \in \mathbb{R}$, $\mathrm{P}(x) \geq 0$.

Q. 2.1 *Show that* P *can be written as*

$$\mathrm{P} = \mathrm{Q} \prod_{1 \leq i \leq p} (\mathrm{X} - \alpha_i)^{\beta_i}$$

where Q *is a polynomial that does not change signs over* $\mathbb{R}$, α_i *are distinct real values and* β_i *are positive integers.*

Q. 2.2 *Show that all* β_i *are even.*

Q. 2.3 *Show that there exist two polynomials* A *and* B *with real coefficients such that* $\mathrm{P} = \mathrm{A}^2 + \mathrm{B}^2$.

Exercise 3 Let X and Y be 2 independent integer-valued random variables. X follows a Bernoulli law of parameter $p \in [0,1]$ and Y follows a Poisson law of parameter $\lambda > 0$.

Let Z be defined as

$$\mathrm{Z} = \begin{cases} 0 \text{ if } \mathrm{X} = 0, \\ \mathrm{Y} \text{ if } \mathrm{X} = 1. \end{cases}$$

Q. 3.1 *What is the law of* Z *?*

Q. 3.2 *Prove that the expected value of* Z *exists and give its value.*

Q. 3.3 *Prove that the variance of* Z *exists and give its value.*

Q. 3.4 *Compute* $\mathrm{P}(\mathrm{X} = 1 | \mathrm{Z} = 0)$.

Exercise 4 Let $f : [0, +\infty) \longrightarrow [0, +\infty)$ be a continuous and non-increasing function such that

$$\int_0^{+\infty} f(t) \mathrm{d}t$$

is convergent.

Q. 4.1 *Let* $h > 0$. *Is the series of general term* $f(nh)$ *convergent ? Prove your answer.*

Q. 4.3 *Let* $a > 0$. *Prove that the integral*

$$\mathrm{I}(a) = \int_{\mathbb{R}} e^{-ax^2} \mathrm{d}x$$

is convergent.

Q. 4.4 *Compute the value of* $\mathrm{I}(1)$.

Q. 4.5 *Prove that* $\mathrm{I} : a \mapsto \mathrm{I}(a)$ *is differentiable.*

Q. 4.6 *Compute the value of*

$$\lim_{t \to 1,\, t<1} \sqrt{1-t} \sum_{n \geq 1} t^{n^2}.$$