CentraleSupélec
Accelerated Engineering Degree Programme
Master's level

September 2018

Physics

2 hours

All answers must be carefully explained and all arguments must be rigorously given.

A bad quality of the hand-writing may hamper the corrector's understanding and, thereby, impact your grade.

No document and no calculator are allowed. Every numerical value can and must be estimated when asked. A standard -non scientific- language dictionary is authorized. Please make surve to have it checked by the staff.

List of universal constants:

- Mass of the electron: $m_e = 10^{-30}\,\mathrm{kg}$
- Boltzmann's constant: $k_\mathrm{B} = 1.4\,10^{-23}\,\mathrm{m^2.kg.s^{-2}.K^{-1}}$
- Reduced Planck's constant: $\hbar = 10^{-34}\,\mathrm{J.s}$
- Speed of light: $c = 3\,10^{-8}\,\mathrm{m.s^{-1}}$
- Elementary charge: $q = 1.6\,10^{-19}\,\mathrm{C}$

Formula:

$$\int_{-\alpha}^{\alpha} x\cos(ax)\sin(bx)\,\mathrm{d}x = \left(\frac{1}{b+a}\right)^2 \sin[(b+a)\alpha] + \left(\frac{1}{b+a}\right)^2 \sin[(b-a)\alpha] - \frac{\alpha}{b+a}\cos[(b+a)\alpha] - \frac{\alpha}{b-a}\cos[(b-a)\alpha]$$

1 Principles of inter-sub-band photodetection

Today's most efficient infrared photodetectors relie on electronic transitions between two quantized states of the conduction band of semi-conductors.

Let us first consider an electron (mass m_e) in an infinite quantum well of width l which is less than $100\,\mu\mathrm{m}$, centred on $x = 0$. The potential energy is zero inside the well.

At thermal equilibrium, the electron has an energy equal to $\mathrm{E} = \frac{3}{2}k_\mathrm{B}\mathrm{T}$.

Q. 1.1 *Give an estimate of the de Broglie wavelength at* $T = 77\,K$.

Q. 1.2 *Explain why this system necessitates a quantum treatment.*

Q. 1.3 *Give the equation that any wave function* $\psi(x,t)$ *of the electron must respect inside the well.*

We will be looking for wave functions under the form: $\psi(x,t) = \phi(x) \times f(t)$.

Q. 1.4 *Give the general form of* $f(t)$.

Q. 1.5 *(This question requires more time than the others.)*
Show that

$$\phi(x) = \phi_n(x) = \left| \begin{array}{l} (-1)^{\frac{n-1}{2}} \mathrm{N}\cos\left(\frac{n\pi}{l}x\right), \text{if n is odd;} \\ (-1)^{\frac{n}{2}} \mathrm{N}\sin\left(\frac{n\pi}{l}x\right), \text{if n is even} \end{array} \right.$$

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Q. 1.6 *Give the value of* N.

Q. 1.7 *Give the expression of the energy associated to a wave function* $\psi(x,t) = \phi_n(x) \times f(t)$ *as a function of* l, n *and* m_e.

Q. 1.8 *Give a numerical value of* l *so that an electron can be excited from the state* $n = 1$ *to* $n = 2$ *when absorbing a photon of wavelength* $\lambda = 4\,\mu\mathrm{m}$.

From now on, we will consider a finite well, of depth $V_0 > 0$ such that it allows only two electronic bound states. To make the calculations easier, we will artificially assume that for these two levels the wave functions and their energies E_1 and E_2 are the same as in the infinite case.

In addition, an electric field $\vec{\mathscr{E}}$ is applied to the structure along $\hat{x}$, providing an extra energy $W = -q\mathscr{E}x$ to the electron and yielding to new energies E'_1 and E'_2.

Q. 1.9 *Give a graphical representation of the new total potential to which the electron is now subjected.*

If $W(x)$ is always small compared to the initial energies $|E_1 - E_2|$, the modified energies and wave functions can be approximated by:

at the first order of approximation,

$$E'_n \approx E_n + \langle n|\hat{W}|n\rangle \tag{1}$$

$$|n\rangle' \approx |n\rangle + \sum_{k\neq n} \frac{\langle k|\hat{W}|n\rangle}{E_n - E_k} |k\rangle \tag{2}$$

at the second order of approximation,

$$\mathrm{E}'_n \approx \mathrm{E}_n + \langle n|\hat{\mathrm{W}}|n\rangle + \sum_{k\neq n} \frac{\left|\langle k|\hat{\mathrm{W}}|n\rangle\right|^2}{\mathrm{E}_n - \mathrm{E}_k}. \tag{3}$$

When using the previous formulæ we will limit ourselves to the only two bound states, that are $n = 1, 2$.

Q. 1.10 *Give a numerical estimate of the typical value of* $\mathscr{E}$ *in* $\mathrm{V.\mu m^{-1}}$ *so that this approximation holds.*

Q. 1.11 *Show that the modification in the energies is at least a second order phenomenon.*

Q. 1.12 *Compute* E'_1 *and* E'_2.

Q. 1.13 *Give the smallest value of* V_0 *as a function of* q, $\mathscr{E}$ *and* l *and the other constants of the system to ensure to always have two electronic levels inside the well.*

Q. 1.14 *Compute and draw the modified wave functions.*

Q. 1.15 *Comment on the shape of the modified wave functions.*

Q. 1.16 *Explain how the application of the electric field creates a current when a photon is absorbed inside the well.*

2 Sparkling curtains

When at night a street lamp is observed through a light curtain, rather than seeing a circular pattern one observes a series a shiny points distributed on a cross. To simplify the modelling of the scene represented on the left side of figure 1, we will use a one dimensional approach. We will suppose that the curtain is made of opaque threads spaced from a distance d (see figure 1, right). We suppose that the street lamp and the observer are both far enough from the curtain so that the rays are considered to be parallel.

Let θ_i be the angle with which the rays of the street lamp are incident on the curtain, and θ_d the one with which they are diffracted toward the observer. We suppose that the street lamp has a wavelength λ.

Q. 2.1 *Express the optical paths differences* δ_1 *and* δ_2 *between two rays passing between to successive threads as a function of* θ_i, θ_d *and* d.

Q. 2.2 *Deduce the phase difference between these two rays as a function of* d *and* λ. *Give a condition so that they constructively interfere at infinity and prove the formula:* $\sin\theta_d = \sin\theta_i + p\frac{\lambda}{d}$ *with* $p \in \mathbb{Z}$.

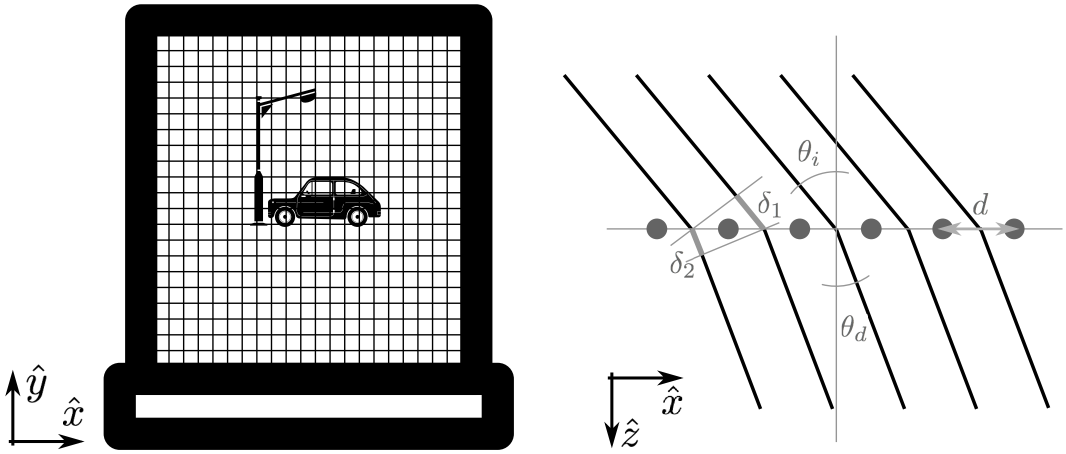

Figure 1: Left: observed scene. Right: one dimensional modelling of the lights rays trajectories through the curtain.

Let us suppose that the street lamp and the observer are aligned along an axis orthogonal to the curtain (axis $\hat{z}$), in such a way that the rays coming from the lamp are at normal incidence on the curtain. This condition, in addition to that of large distances, enables the use of small angles approximation.

Q. 2.3 *Show that this yields to constructive interferences at angles* $\theta_d^p = p\frac{\lambda}{d}$ *with* $p \in \mathbb{Z}$.

Due to the finite size of the source, only five spots can be observed. In other words, p can only take the following values: $p \in \{-2, -1, 0, 1, 2\}$.

The eye of the observer is modelled by a lens (corresponding to the cornea) of focal distance $f = 2\,\text{cm}$, forming images on the retina placed in its focal plane (Figure 2). The retina is considered to be flat due the small angles approximation.

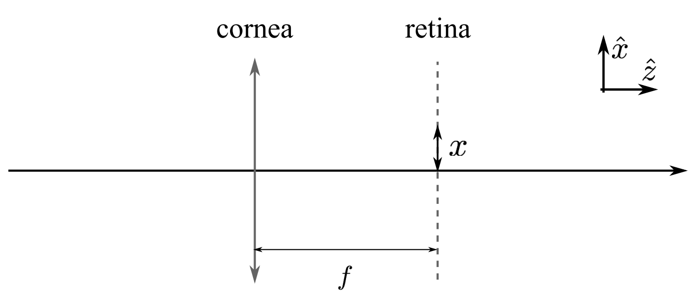

Figure 2: Eye schematics.

Q. 2.4 *Compute the abscissæ* x^p *of the images of the spots on the retina.*

Knowing the x^p values, hence the angles θ_d^p, would allow finding the distance d between two threads of the curtain. However, in reality, it is quite difficult to measure distances on a retina. Comparisons with other objects (e.g. a ruler) must be used. To do so, a car of length $l = 2.4\,\mathrm{m}$ is conveniently parked under, and also centred on, the street lamp. The whole scene is at a distance $\mathrm{L} = 20\,\mathrm{m}$ of the observer.

This latter slightly adapt his position so that the spots $p = \pm 2$ are superimposed to the car extremities. The lamp uses a traditional sodium lamp of wavelength $\lambda = 600\,\mathrm{nm}$.

Q. 2.5 *Give the numerical value of* θ_d^2 *and deduce the inter-thread distance* d.

The same experiment is intended the day after. However, the sodium lamp have been replaced by a white LED emitting from $\lambda_{\mathrm{min.}} = 400\,\mathrm{nm}$ to $\lambda_{\mathrm{max.}} = 800\,\mathrm{nm}$.

Q. 2.6 *Estimate the angles* $\theta_d^p\big|_{\mathrm{min.}}$ *and* $\theta_d^p\big|_{\mathrm{max.}}$ *for* $p = 1, 2$ *and* 3*, then conclude.*